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On Solutions of Liouville's Equation

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1. INTRODUCTION

It is well known that a large class of nonlinear evolution equations can be solved by the inverse scattering method (see [1, 5, 6, 16]). Typical examples are the Korteweg–de Vries (KdV) equation, the modified KdV equation, the nonlinear Schrödinger equation, and the sine–Gordon equation. In [8], Lax first showed a general method for constructing equations which can be solved by the inverse scattering method: Let $\{L(t)\}$ and $\{B(t)\}$ be a pair of one-parameter families of linear operators satisfying the operator equation

$$L_t = [B, L] = BL - LB, \tag{1.1}$$

where the subscript t denotes the differentiation with respect to t. Then, members of $\{L(t)\}$ are similar to each other, the eigenvalue of $L(t)$ is independent of t and the corresponding eigenfunction ψ evolves in t according to the equation

$$\psi_t = B\psi. \tag{1.2}$$

In concrete representation, operators $L(t)$, $B(t)$ are integral or differential operators and the operator equation (1.1) is reduced to a system of nonlinear evolution equations for the coefficients of $L(t)$ and $B(t)$. Other approaches for constructing nonlinear evolution equations which are solved by the inverse scattering method have been developed by Ablowitz *et al.* [1], Wahlquist and Estabrook [15], and Zakharov and Shabat [16]. On the other hand, the Bäcklund transformations are found to be very useful for constructing solutions of nonlinear evolution equations and deeply connected with the inverse scattering method (see [3, 7, 14]). Chen showed a general way of obtaining the Bäcklund transformation from Lax's formalism.

It was shown in Forsyth [4, Vol. 6, p. 441] that Liouville's equation $s = \exp u$ has the Bäcklund transformation

$$p + p' = 2^{1/2} \exp[\tfrac{1}{2}(u - u')], \tag{1.3}$$

$$q - q' = 2^{1/2} \exp[\tfrac{1}{2}(u + u')], \tag{1.4}$$

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where u satisfies $s = \exp u$ and u' satisfies $s' = 0$. Here notations $p = u_\xi$, $q = u_\eta$, $s = u_{\xi\eta}$ have been employed. In [13] the present author found a relationship between Liouville's equation and the nonlinear eigenvalue problem (3.3) stated below. In this paper our aim is to demonstrate that Liouville's equation can be treated by Lax's formalism and has an infinite set of conservation laws. It will be shown that the auto–Bäcklund transformation for Liouville's equation can also be obtained from Lax's method (see [7]).

2. Liouville's Equation

Consider Liouville's equations of the form

$$u_{tt} - u_{xx} + Ke^u = 0, \tag{2.1}$$

and of the form

$$u_{tt} + u_{xx} + Ke^u = 0, \tag{2.2}$$

where K denote (complex or real) constants. Equation (2.1) can be rewritten in terms of new variables

$$\xi = x + t, \qquad \eta = x - t \tag{2.3}$$

as

$$u_{\xi\eta} = \frac{K}{4} e^u. \tag*{(2.1)$'$}$$

Introducing the complex variables

$$z = x + it, \qquad \bar{z} = x - it, \tag{2.4}$$

we can write (2.2) in the form

$$u_{z\bar{z}} + \frac{K}{4} e^u = 0. \tag*{(2.2)$'$}$$

We now show that (2.1) and (2.2) can be treated by the Lax formalism. Introduce two 3×3 matrix differential operators

$$L(t) = \begin{pmatrix} 0 & \dfrac{\partial}{\partial x} + v(x, t) & 0 \\ -\dfrac{\partial}{\partial x} + v(x, t) & 0 & w(x, t) \\ 0 & w(x, t) & 0 \end{pmatrix} \tag{2.5}$$

and

$$B(t) = \begin{pmatrix} \dfrac{\partial}{\partial x} & 0 & -2w(x,t) \\ 0 & \dfrac{\partial}{\partial x} & 0 \\ 2w(x,t) & 0 & -\dfrac{\partial}{\partial x} \end{pmatrix}. \tag{2.6}$$

It is easily verified by direct substitution that the Lax operator equations

$$L_t = [B, L] \tag{2.7}$$

and

$$L_t = [iB, L] \tag{2.8}$$

are identical with the systems

$$\begin{aligned} v_t - v_x + 2w^2 &= 0, \\ w_t + w_x - 2vw &= 0 \end{aligned} \tag{2.9}$$

and

$$\begin{aligned} v_t - iv_x + 2iw^2 &= 0, \\ w_t + iw_x - 2ivw &= 0, \end{aligned} \tag{2.10}$$

respectively. If we take

$$v = (u_t + u_x)/4, \qquad w = K^{1/2}e^{u/2}/2(2)^{1/2}, \tag{2.11}$$

(2.9) reduces to (2.1). If we take

$$v = (u_t + u_x)/4i, \qquad w = -K^{1/2}e^{u/2}/2(2)^{1/2}, \tag{2.12}$$

(2.10) reduces to (2.2).

Remark 1. Eliminating w from (2.9) and introducing new independent variables (2.3), we get

$$v_{\xi\eta} - (v^2)_\eta = 0. \tag{2.13}$$

Integration of (2.13) gives the Riccati equation

$$v_\xi - v^2 = f(\xi), \tag{2.14}$$

where $f(\xi)$ is an arbitrary function. Solving (2.14), we obtain the general solution of (2.1)′ given by

$$e^{u(\xi,\eta)} = \frac{8}{K}\frac{F'(\xi)\,G'(\eta)}{[F(\xi) + G(\eta)]^2}, \tag{2.15}$$

where $F(\xi)$ and $G(\eta)$ are arbitrary functions. The function $f(\xi)$ is simply related to $F(\xi)$, namely

$$2f(\xi) = \{F, \xi\}, \tag{2.16}$$

where $\{\cdot, \cdot\}$ denotes the Schwarzian derivative. This derivation of the general solution of (2.1)′ was first found by Liouville [11] (see also Bianchi [2]). An analogous result holds valid for (2.2)′.

Below we consider only smooth real solutions of the initial value problems for (2.1) and (2.9) with appropriate asymptotic boundary conditions as $|x| \to \infty$ or with periodic boundary condition.

Remark 2. The existence and uniqueness of smooth real solutions of the initial value problems for (2.1) and (2.9) under appropriate boundary conditions may be established by the usual methods since (2.1) and (2.9) are semilinear hyperbolic equations. (see [10, 12]).

3. Associated Eigenvalue Problems

Consider the eigenvalue problem

$$L(t)\,\psi = \lambda\psi. \tag{3.1}$$

Let $\psi = \mathrm{col}(\psi_1, \psi_1, \psi_3)$, where col denotes a column vector. Then, we have

$$\begin{aligned} -(\psi_1)_x + v(x,t)\,\psi_1 + w(x,t)\,\psi_3 &= \lambda\psi_2, \\ (\psi_2)_x + v(x,t)\,\psi_2 &= \lambda\psi_1, \\ w(x,t)\,\psi_2 &= \lambda\psi_3. \end{aligned} \tag{3.2}$$

If $\lambda \neq 0$, we can eliminate ψ_3 from (3.2):

$$\begin{aligned} -(\psi_1)_x + v(x,t)\,\psi_1 + \frac{1}{\lambda}\,w^2(x,t)\,\psi_2 &= \lambda\psi_2, \\ (\psi_2)_x + v(x,t)\,\psi_2 &= \lambda\psi_1. \end{aligned} \tag{3.3}$$

It is a nonlinear eigenvalue problem. The construction of the theory of the inverse scattering problem may be difficult. But, eliminating ψ_1, we get the one-dimensional Schrödinger (or Hill's) equation:

$$(\psi_2)_{xx} + (\lambda^2 - U(x,t))\,\psi_2 = 0, \tag{3.4}$$

where

$$U(x,t) = v^2(x,t) + w^2(x,t) - v_x(x,t). \tag{3.5}$$

The spectral theory and the inverse scattering theory for (3.4) with real potential U have been studied extensively under the asymptotic conditions:

$$\begin{aligned} U(x) &\to a \quad \text{as} \quad x \to \infty, \\ &\to b \quad \text{as} \quad x \to -\infty, \end{aligned} \tag{3.6}$$

where a and b are real constants, or the periodic condition with period T:

$$U(x + T) = U(x), \qquad -\infty < x < \infty. \tag{3.7}$$

Lax's equation (2.7) yeilds

THEOREM 1. *If $v(x, t)$ and $w(x, t)$ change according to the system* (2.9), *the eigenvalues of* (3.4) *are independent of t.*

THEOREM 2. *If $u(x, t)$ changes according to Liouville's equation* (2.1), *then the eigenvalues of* (3.4) *with*

$$U(x, t) = \frac{1}{16}(u_t + u_x)^2 + \frac{K}{8} e^u - \frac{1}{4}(u_{xt} + u_{xx}) \tag{3.8}$$

are independent of t.

Remark 3. The inverse scattering theory for (3.4) with (3.5) or (3.8) seems to be less useful for solving the system (2.9) or Liouville's equation (2.1).

Remark 4. Under the transformation $t \to -t$, (2.1) is invariant. Hence, the assertion of Theorem 2 holds for the eigenvalue problem (3.4) replacing $U(x, t)$ by

$$\tilde{U}(x, t) = \frac{1}{16}(u_t - u_x)^2 + \frac{K}{8} e^u + \frac{1}{4}(u_{xt} - u_{xx}). \tag{3.9}$$

Remark 5. Let $u(x, t)$ evolve according to (2.1). Straightforward calculation gives

$$U_t - U_x = 0 \tag{3.10}$$

and

$$\tilde{U}_t + \tilde{U}_x = 0, \tag{3.11}$$

which also imply the assertions of Theorem 2 and Remark 4. Furthermore, (3.10) and (3.11) imply that the Liouville's equation has an infinite family of conservation laws of the forms:

$$(U^n)_t \pm - (U^n)_x = 0, \qquad n = 1, 2,..., \tag{3.12}$$

or

$$(\tilde{U}^n)_t + \tilde{U}^n)_x = 0, \qquad n = 1, 2,.... \tag{3.13}$$

Remark 6. In view of Remarks 1 and 5, it is seen that

$$U(x, t) = \tfrac{1}{2}\{F, x + t\}. \tag{3.14}$$

Analogously we obtain

$$\tilde{U}(x, t) = \tfrac{1}{2}\{G, x - t\}. \tag{3.15}$$

It is well known that the Schwarzian derivative is invariant under linear fractional transformation acting on its first argument:

$$\{F, x\} = \left\{\frac{aF + b}{cF + d}, x\right\}, \tag{3.16}$$

where a, b, c, d are constants ($ad - bc = 1$). Hence, for the same potential U (or $\tilde{U}$) infinitely many different F's (or G's) satisfy (3.14) (or (3.15)).

4. Bäcklund Transformation and Conservation Laws

Lax's formalism asserts that the eigenfunction $= \mathrm{col}(\psi_1, \psi_2, \psi_3)$ of (3.1) (or equivalently, of (3.2)) evolves in t according to (1.2) with (2.6), namely,

$$\begin{aligned}(\psi_1)_t &= (\psi_1)_x - 2w(x, t)\,\psi_3,\\ (\psi_2)_t &= (\psi_2)_x,\\ (\psi_3)_t &= -(\psi_3)_x + 2w(x, t)\,\psi_1,\end{aligned} \tag{4.1}$$

if v and w change according to the system (2.9). Letting $\phi = \psi_1/\psi_2$, we get two Riccati equations from (3.3) and (4.1):

$$\phi_x = 2v\phi - \lambda\phi^2 + w^2\lambda^{-1} - \lambda, \tag{4.2}$$

$$\phi_t = \phi_x - 2w^2\lambda^{-1}, \tag{4.3}$$

from which it follows that

$$\begin{aligned}\phi_t - \phi_x &= -2w^2\lambda^{-1},\\ \phi_t + \phi_x &= 4v\phi - 2\lambda\phi^2 - 2\lambda.\end{aligned}$$

Transformation (2.3) gives

$$\phi_\eta = w^2\lambda^{-1}, \tag{4.4}$$

$$\phi_\xi = 2v\phi - \lambda\phi^2 - \lambda. \tag{4.5}$$

If we take (2.11) and put

$$h(\xi, \eta) = u(\xi, \eta) - 2\log(\phi(\xi, \eta)/a), \tag{4.6}$$

where a is an arbitrary constant, from (4.4) and (4.5) we have

$$\begin{aligned} u_\xi + h_\xi &= 2\lambda a e^{(u-h)/2} - \frac{\lambda}{a} e^{-(u-h)/2}, \\ u_\eta - h_\eta &= \frac{K}{4a\lambda} e^{(u+h)/2}, \end{aligned} \tag{4.7}$$

where h satisfies

$$h_{\xi\eta} = \frac{K}{4a^2} e^h. \tag{4.8}$$

Equation (4.7) with $a = 1$ is an auto–Bäcklund transformation for Liouville's equation.

We now derive an infinite set of conservation laws from the associated eigenvalue problem (3.2) with (4.1). Put $\Gamma = \lambda\phi - v$, from (4.2) and (4.3) we have

$$\Gamma_x = v^2 + w^2 - v_x - \Gamma^2 - \lambda^2, \tag{4.9}$$

$$\Gamma_t - \Gamma_x = 0. \tag{4.10}$$

We may confirm that $\Gamma(x, t, \lambda)$ has the asymptotic representation of the form

$$\Gamma(x, t, \lambda) \sim \sum_{n=0}^{\infty} \Gamma_n(x, t)\, \lambda^{1-n},$$

where the coefficients $\Gamma_n(x, t)$ are determined by the following recursion formula:

$$\begin{aligned} \Gamma_0 &= i, \\ (\Gamma_n)_x &= (v^2 + w^2 - v_x)\, \delta_{n,1} - \sum_{k=0}^{n+1} \Gamma_{n+1-k}\Gamma_k \qquad (n = 0, 1, 2, \ldots). \end{aligned} \tag{4.12}$$

From (4.10) we have an infinite set of conservation laws:

$$(\Gamma_n)_t - (\Gamma_n)_x = 0, \qquad n = 1, 2, \ldots. \tag{4.13}$$

The first four Γ_n's are

$$\begin{aligned} \Gamma_1 &= 0, \\ \Gamma_2 &= \frac{1}{2i}(v^2 + w^2 - v_x) \\ &= \frac{1}{32i}(u_t + u_x)^2 + \frac{K}{16i} e^u - \frac{1}{8i}(u_{xt} + u_{xx}), \\ \Gamma_3 &= -\frac{1}{2i}(\Gamma_2)_x, \\ \Gamma_4 &= -\frac{1}{2i}\Gamma_2^2 - \frac{1}{4}(\Gamma_2)_x. \end{aligned}$$

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